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FORCED CONVECTION FROM A CYLINDER AT MODERATE REYNOLDS NUMBERS

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Abstract—Forced convection from circular cylinders in crossflow at moderate Reynolds numbers shows departures from flow and heat transfer predicted by boundary-layer analysis because of boundary-layer curvature. In the forward stagnation region the velocity gradient and its rate of change in the vicinity of the surface increase as Re decreases. At large Prandtl numbers the change in heat transfer is associated with the increase in wall velocity gradient, and is greater than expected through curvature of the thermal boundary layer itself. For Pr below about 0·1, effects of curvature of the thermal boundary layer exceed effects of change in wall velocity gradient.

NOMENCLATURE

f_i, stream functions;
F_i, dimensionless stream functions;
G_i, dimensionless temperature profiles;
R, cylinder radius;
r, radius at a point in the fluid surrounding cylinder;
ζ, dimensionless radial coordinate;
η, $\zeta - Re_R^{1/2}$;
θ, dimensionless temperature;
ν, kinematic viscosity;
ϕ, azimuthal angle, measured from front stagnation point;
ψ, stream function;
Nu_R, Nusselt number based on cylinder radius R;
Pr, Prandtl number;
Re_R, Reynolds number based on cylinder radius R.

INTRODUCTION

FORCED convection from circular cylinders at moderate Reynolds numbers exhibits considerable departures from the flow and heat transfer predicted by boundary-layer analysis. These departures are caused predominantly by the boundary-layer curvature which is ignored in the asymptotic ($Re \to \infty$) analysis, these being cast in rectangular curvilinear coordinates, e.g. Blasius [1] and Goertler [2]. The departure is most pronounced for very small Prandtl numbers at any finite Reynolds number, because the thermal boundary layer becomes increasingly thicker than the viscous boundary layer (and is therefore affected by its curvature more) as the Prandtl number is reduced. For boundary-layer analysis of convection round the forward portion of a cylinder, the expansion of Merk [3] is convenient to use, and the convergence scheme of Nachtsheim and Swigert [4] permits solution for any Prandtl number of interest through use of a digital computer.

Some estimates of convection at moderate Reynolds numbers have been made, including Van Dyke's second-order calculations using expansion methods [5]. Some of the interest in second-order effects has been stimulated by observations in hypersonic experiments, as reviewed by Lewis [6]. However, for the classical problem in incompressible flow, the proper formulation of the problem requires solution of the two-dimensional Navier–Stokes and energy equations expressed most conveniently in cylindrical coordinates. These give rise to a momentum equation of higher order than that which comes from boundary-layer theory. This complicates computation of solutions because it requires a larger number of boundary values at the cylinder surface to be found.

ANALYSIS

The formulation of the problem for which solutions are given here is rather similar to that used for a study on the effect of boundary-layer curvature in natural convection over a horizontal cylinder [7]. In the present case there is no buoyancy term but instead a steady flow transverse to the cylinder and which is uniform infinitely upstream of the cylinder. In common with the natural convection problem the first pair of differential equations derived through application of the Blasius-style or the Goertler-style transformations (the original equations here being written in cylindrical coordinates) turns out to be the same for both transformations; for the front stagnation region the solution of this first pair of equations is dominant. The choice of the azimuthal series is open to other possibilities. Besides the series of Blasius and of Goertler, each of these series being designed for use in boundary layer problems ($Re \to \infty$), one finds series such as that used for flow around a cylinder in the Reynolds number range $O(1)$ by Underwood [8]. Underwood's series expansion for the stream function was

$$\psi = f_1(r)\sin\phi + f_2(r)\sin 2\phi + \dots \tag{1}$$

an expansion which yielded equations for solution on a series-truncation basis. A particular advantage which arose at $Re = O(1)$ was that the sequence of differential equations could be arranged in a set for each of the f_i with a group of linear terms in derivatives of f_i making up the left-hand-side, the right-hand-side being a function of all f_i's and Re. This sequence was solved iteratively through series truncation. In the present case solutions are sought in the range $\infty > Re > 900$, and series appropriate for boundary-layer-like problems seem to be a useful starting point: presumably, as Re is progressively reduced the (truncated) series will need to be taken to an increasing number of terms to maintain a constant precision of solution and at some Reynolds number a switch to an expansion such as (1) would provide a more efficient means of solution because the number of terms in the series needed would be smaller. Thus, by use of the Blasius-style transformations for the radial coordinate

$$\zeta = Re_R^{1/2} r/R, \tag{2}$$

the stream function (an azimuthal series)

$$\psi = \nu Re_R^{1/2}\{\phi F_1(\zeta) + \phi^3 F_3(\zeta)\dots\} \tag{3}$$

and the dimensionless temperature

$$\theta = G_0(\zeta) + \phi^2 G_2(\zeta) + \dots \tag{4}$$

the Navier–Stokes and energy equations reduce to sets of terms which are coefficients of an infinite power series of the azimuthal coordinate ϕ; each set which is separately a coefficient of a power of ϕ can be put equal to zero. It is found that the set for each power of ϕ, while dominated by terms for the corresponding F_i, contain some terms for other F_i as well. This implies that the process of solution is not simply a matter of marching through equations of ascending order in F_i's, using only solutions for lower values of the series index (i), as can be done in the boundary-layer limit; instead, an iterative process appears to be needed because even the first equation (obtained by putting the coefficient of ϕ equal to zero) contains terms in F_3 as well as F_1. A continuous process of revising solutions for F_1, F_3, etc. seems to be called for, in principle, as higher orders of F_i are numerically investigated. The very first solution which can be sought is one for the first pair of equations (momentum and energy) from which terms in higher orders of F_i have been deleted. Terms in the momentum equation, for example, can be arranged in sub-sets each of different order in $(1/Re)^{1/2}$; the first such sub-set of order (1), is

$$F_1^{\mathrm{IV}} + \frac{Re^{1/2}}{\zeta} F_1 F_1''' - \frac{Re^{1/2}}{\zeta} F_1' F_1''$$

and this is the sub-set which yields the boundary-layer limit when integrated once; the sub-set of order $(1/Re)^{1/2}$ is

$$\left(\frac{2}{\zeta}\right) F_1''' + \frac{Re^{1/2}}{\zeta^2} F_1 F_1'' - \frac{Re^{1/2}}{\zeta^2}(F_1')^2;$$

this sub-set represents the first-order correction for finite Re effects and does not contain F_3 in any form: the sub-set of order $(1/Re)$ is

$$-\frac{1}{\zeta^2} F_1'' - \frac{Re^{1/2}}{\zeta^3} F_1 F_1' - 6\frac{Re^{1/2}}{\zeta^3} F_1' F_3 + 6\frac{Re^{1/2}}{\zeta^3} F_1 F_3' + \frac{12}{\zeta^3} F_3''.$$

This sub-set is the second-order correction for finite Re effects and does contain terms involving F_3. By deleting terms in F_3, it is to this order that solution of the equation truncated of terms in F_3 will be approximate. It is possible to estimate the magnitudes of terms here because at large Re, F_1 and F_3 will still be fairly close to values determined from the Blasius series solutions [9]. From this it seems that the terms in $F_1' F_3$ and $F_1 F_3'$ will nearly mutually cancel, and that omission of the term in F_3'' has the greatest effect in making the solution of the momentum equation approximate. The situation with the energy equation is simpler; no terms in G_2 etc. appear in the energy equation for $(\phi)^0$, but the F_1 used in that equation is of course an approximation in the sense discussed in detail above. Understanding that the solution of equations from which terms in F_3 etc. have been deleted is the zeroth iterate at finite Re in a series truncation, the truncated equations for F_1 and G_0 can be written (dropping the subscripts) as

$$\begin{aligned} F^{\mathrm{IV}} &+ (2/\zeta)F''' - (1/\zeta^2)F'' + (1/\zeta^3)F' \\ &- (Re^{1/2}/\zeta^2)(F')^2 - (Re^{1/2}/\zeta)F'F'' - (Re^{1/2}/\zeta^3)FF' \\ &+ (Re^{1/2}/\zeta^2)FF'' + (Re^{1/2}/\zeta)FF''' = 0 \end{aligned} \tag{5}$$

and

$$G'' + (1/\zeta)G' + (Re_R^{1/2}/\zeta)PrFG' = 0 \tag{6}$$

with the boundary conditions, after corresponding transformations, being written

$$\begin{aligned} &F = 0,\ F' = 0,\ G = 1 \quad \text{at} \quad r = R \text{ i.e. } \zeta = Re_R^{1/2} \\ &F' \to 1,\ F'' \to 0,\ G \to 0 \quad \text{as} \quad r \to \infty \text{ i.e. } \zeta \to \infty. \end{aligned} \tag{7}$$

These equations reduce to those for a boundary layer as $Re_R \to \infty$ if the transformation $\eta = \zeta - Re_R^{1/2}$ is used.

In the process of solution, the three boundary values F'', F''' and G' at $r = R$, i.e. $\zeta = Re_R^{1/2}$, must be found. This can be accomplished by iteration seeking values for F'' and F''' at $\zeta = Re_R^{1/2}$ which simultaneously make $F' \to 1$ and $F'' \to 0$ at sufficiently large value of ζ in momentum equation. After the momentum equation has been solved for a particular Reynolds number, the energy equation can be solved for a particular Prandtl number by seeking G' at $\zeta = Re_R^{1/2}$ to satisfy the asymptotic condition, $G \to 0$ as $\zeta \to \infty$. Integration of equations (4) and (5) was carried out by fourth-order Runge–Kutta and Adams–Moulton programs and the Nachtsheim and Swigert convergence scheme was used.

DISCUSSION

The results of the calculations quantify the expected trends, Table 1. The effect of finite boundary-layer curvature on structure in the viscous boundary layer is to increase the velocity gradient (F'') and its rate of change (F''') in the vicinity of the wall, compared with the boundary-layer asymptotic values, as Re is decreased. At large Prandtl numbers the thermal boundary layer is small compared with the viscous boundary layer, and its quantitative structure is dominantly determined by the velocity gradient at the wall; thus the effect of finite Reynolds numbers is felt dominantly through the shift in F''. The effect is greater than would be accountable through curvature of the thermal boundary layer by itself. For the asymptotic boundary-layer case, values are indicated by the dashed line. At very small Prandtl numbers the thermal boundary layer is larger than the viscous boundary layer, and the effects of curvature of the thermal boundary layer itself exceeds the effect of the shift in F''. These two cases can be seen in Fig. 1 where $Nu_R/Re_R^{1/2}Pr^{1/3}$ is plotted as a function of Pr. The numerical effects of the two Prandtl number extremes can be seen; what is not obvious before numerical evaluation is the order of magnitude of Prandtl number which divides the two extremes. From Fig. 1, this is about $Pr = 0{\cdot}1$. The distinction between the two extremes can be seen in temperature gradient profiles from Prandtl numbers typical of the ranges, Fig. 2; at $Pr = 0{\cdot}005$ there is a large change in $G'(0)$ from $Re = 10\,000$ to $Re = 900$, an effect of thermal boundary-layer curvature itself,

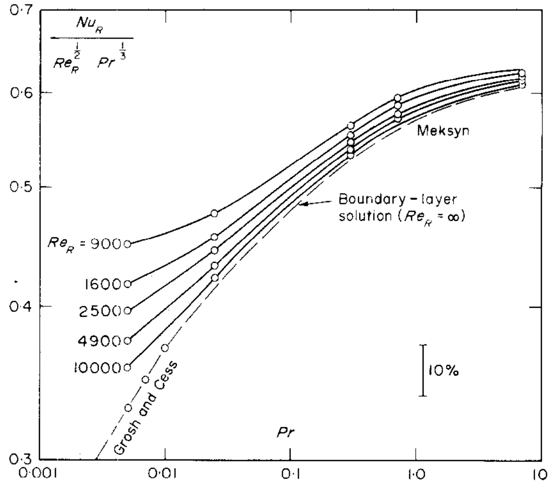

FIG. 1. Effect of finite Reynolds numbers on heat transfer at the forward stagnation point of a cylinder over a range of Prandtl numbers; the boundary-layer limit solutions are shown as a dashed line.

Table 1. Boundary values at $r = R$

	$Pr =$		0·005	0·025	0·3	0·72	7·0
Re_R	F''	$-F'''$	$-G'$	$-G'$	$-G'$	$-G'$	$-G'$
∞	1·2326	1·0000	(0·0506)	(0·126)	(0·348)	(0·499)	(1·177)
10^4	1·267	1·054	0·0609	0·1232	0·3569	0·5056	1·166
8100	1·271	1·060	0·0617	0·1240	0·3578	0·5066	1·168
6400	1·276	1·068	0·0626	0·1249	0·3589	0·5079	1·170
4900	1·276	1·070	0·0638	0·1261	0·3598	0·5087	1·172
3600	1·283	1·083	0·0653	0·1274	0·3616	0·5108	1·174
2500	1·286	1·091	0·0677	0·1300	0·3636	0·5128	1·176
1600	1·300	1·114	0·0712	0·1334	0·3673	0·5168	1·182
900	1·322	1·154	0·0772	0·1393	0·3734	0·5236	1·193
Bound	0·003	0·005	0·0003	0·0003	0·0003	0·0003	0·001

Values in parentheses are from asymptotic methods. Bottom row gives estimated error bounds of values in columns associated with programmed tolerance in satisfying outer boundary condition. Some fluctuation in the values of $F''(0)$ occur because the program adapted to changes in effective boundary-layer thickness by making finite shifts in the position of the outer edge of the boundary layer, as in [4]; such shifts occurred between $Re_R = 6400$ and 4900 and between 3600 and 2500. The boundary-layer thickness on the η scale falls as $F'(0)$ rises, i.e. as Re decreases.

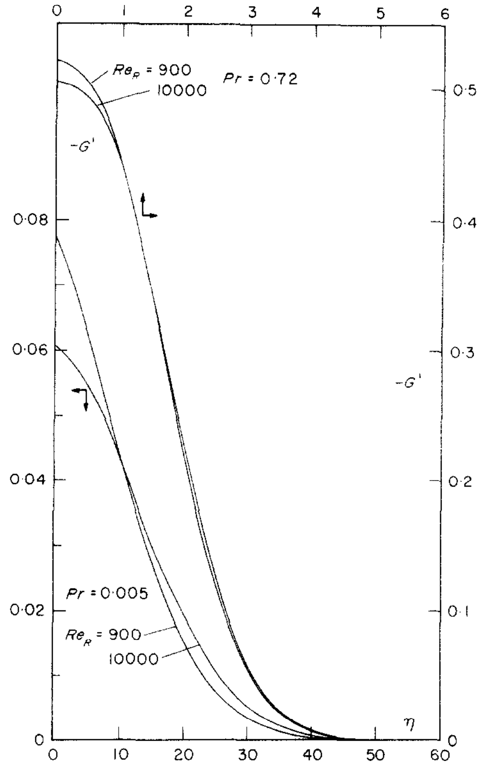

FIG. 2. Temperature gradient profiles at the front stagnation point region of a transverse cylinder at finite Reynolds numbers for two different Prandtl numbers.

but much less at $Pr = 0{\cdot}72$. As expected, for large Reynolds numbers, solutions approach Meksyn's boundary-layer series expansion [10] at high Prandtl numbers and approach Grosh and Cess's inviscid flow solution [11] at low Prandtl numbers. It appears from Fig. 3 that Grosh and Cess's solution slightly underestimates the heat transfer at smaller low-Prandtl numbers and slightly overestimates the heat transfer at larger low-Prandtl numbers in the liquid metal region.

Experimental data on *local* heat transfer in the ranges of Reynolds and Prandtl numbers discussed here is rather scarce. The interferometer data of Eckert and Soehngen [12] straddle the line extrapolated for $Pr = 0{\cdot}72$, Fig. 3. This figure illustrates that the effect of boundary-layer curvature is to make $\partial(\log Nu)/\partial(\log Re) < 0{\cdot}5$ at moderate Re, an effect shown well in data for overall heat transfer from a cylinder in crossflow. As noted previously [13], incorporation of boundary-layer curvature effects can lead to approximations for the heat transfer of the type $Nu = A + BRe^{1/2}$, which have been used quite widely on an empirical basis in the past, e.g. [12, 14].

For the front laminar region, the correlation for air turns out to be [13]

$$Nu_D = 0{\cdot}3737 + 0{\cdot}37(Re_D)^{1/2}.$$

This approximation is within 4 per cent of the computed value within the Reynolds number range considered here.

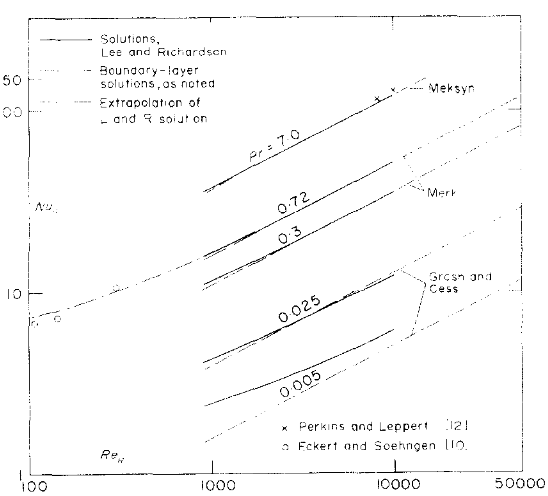

FIG. 3. Effect of Reynolds number on heat transfer at various Prandtl numbers, showing the divergence of the computed solution from the boundary-layer analysis; some experimental data are also shown.

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CONVECTION FORCEE PAR UN CYLINDRE A DES NOMBRES DE REYNOLDS MODERES

Résumé—Aux nombres de Reynolds modérés on constate à cause de la courbure de la couche limite, un écart entre la convection forcée par les cylindres circulaires dans un écoulement frontal et les prédictions de transfert thermique à partir de l'analyse de couche limite. Dans la région du point d'arrêt amont, le gradient de vitesse et son taux de croissance au voisinage de la surface croît lorsque *Re* diminue. Aux grands nombres de Prandtl le changement de transfert thermique est associé à l'accroissement du gradient pariétal de vitesse et il est supérieur à ce qui peut être rapporté à la courbure de la couche limite thermique. Pour *Pr* de l'ordre de 0,1, les effets de la courbure de la couche limite thermique excèdent ceux du gradient pariétal de vitesse.

ERZWUNGENE KONVEKTION AM ZYLINDER BEI KLEINEREN REYNOLDS-ZAHLEN

Zusammenfassung—Erzwungene Konvektion an Zylindern mit kreisförmigem Querschnitt im Kreuzstrom bei kleineren Reynolds-Zahlen zeigt wegen der Grenzschichtkrümmung Abweichungen in der Strömungsform und der Wärmeübertragung von den nach der Grenzschichttheorie bestimmten Werten. Im vorderen Staubereich wächst der Geschwindigkeitsgradient und seine Änderung in Oberflächennähe in dem Maße an, in dem *Re* fällt. Bei hohen Prandtl-Zahlen ist die Änderung der Wärmeübertragung mit dem Anwachsen des Geschwindigkeitsgradienten an der Wand gleichläufig und größer als durch die Krümmung der thermischen Grenzschicht erwartet.

Bei $Pr \lesssim 0{,}1$ übertreffen die Auswirkungen der Krümmung der thermischen Grenzschicht die Wirkung der Änderung im Geschwindigkeitsgradienten an der Wand.

ВЫНУЖДЕННАЯ КОНВЕКЦИЯ ОКОЛО ЦИЛИНДРА ПРИ УМЕРЕННЫХ ЧИСЛАХ РЕЙНОЛЬДСА

Аннотация — При вынужденной конвекции около круглых цилиндров в поперечном потоке для умеренных чисел Рейнольдса вследствие кривизны пограничного слоя наблюдается отличие в переносе импульса и тепла от данных, полученных в приближении пограничного слоя. В передней зоне торможения градиент скорости и вторая производная вблизи поверхности возрастают по мере уменьшения *Re*. При больших числах Прандтля изменение теплопереноса связано с увеличением градиента скорости на стенке, и тепловой поток оказывается большим, чем можно было бы предположить с учётом кривизны теплового пограничного слоя. Для значений *Pr* ниже 0,1 эффект кривизны теплового пограничного слоя превышает эффект изменения градиента скорости на стенке.